Material Models for Nonlinear Deformation of Graphite

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ATJ-S graphite is a transversely isotropic granular composite material with different nonlinear stress-strain behavior under tension loading than under compression loading. A model for nonlinear deformation behavior is extended from initial loading under biaxial tension stresses to initial loading under mixed tension and compression stresses. The principal basis of the model is that the material properties are expressed as a function of the strain energy of an equivalent elastic material. Thus, interaction between the various components of a multiaxial stress state is accounted for. Different moduli in tension and compression are excited and are included in the analysis. Both the Jones weighted compliance matrix (WCM) material model and the Isabekian and Khachatryan restricted compliance matrix (RCM) material model are extended to nonlinear deformation.

Nomenclature

A_i, B_i, C_i, U_{0_i} = constants in *i*th material property equation [Eq. (19)]
E_r, E_z, E_θ = Young's moduli in principal material directions
E^{45} = Young's modulus at 45° to r and z directions
G_{rz} = shear modulus in rz plane
n = constant in energy weighting function [Eq. (22)]
r, z, θ = radial, axial, and circumferential directions [Fig. 1]
U = strain energy function [Eq. (20)]
γ_{rz} = shear strain in rz plane
$\epsilon_r, \epsilon_z, \epsilon_\theta$ = strains in principal material directions
$\nu_{rz}, \nu_{r\theta}, \nu_{z\theta}$ = Poisson's ratios for principal material directions
$\sigma_r, \sigma_z, \sigma_\theta, \tau_{rz}$ = axisymmetric stresses in principal material coordinates

Subscripts

c = compression
t = tension
w = weighted

Superscripts

mn = principal material coordinates
pq = principal stress coordinates

Introduction

ARTIFICIAL graphites are transversely isotropic granular composite materials made in billet form as in Fig. 1. Deformation characteristics of primary importance are nonlinear stress-strain curves, biaxial softening, and different stress-strain behavior under tension loading than under compression loading. Biaxial softening is the development of slightly larger strains in biaxial tension than in uniaxial tension, as shown in Fig. 2, in contradiction to what might be expected on the basis of conventional Poisson effects.[1] In fact, Poisson's ratios for ATJ-S graphite decrease with increasing tensile stress instead of increasing as for metals such as aluminum. Graphites also behave quite differently under tensile stress than under compressive stress as seen in the typical stress-strain curves of Fig. 3. There, the initial slopes (elastic moduli) of the curves are different leading to bilinear characterization in the form of Fig. 4 as an approximation to the actual nonlinear behavior. Also, the Poisson's ratios decrease in tension, but are constant or increase in compression.

The objective of this paper is to present a theory for simultaneously predicting biaxial softening behavior and different nonlinear behavior in tension and compression. This research is an extension of the authors' treatment of softening under biaxial tension[2] and of off-axis uniaxial loading.[3] Both the Jones weighted compliance matrix material model[4] and the Isabekian and Khachatryan restricted compliance matrix material model[5] for materials with different moduli in tension and compression are extended to nonlinear deformation of axisymmetric bodies under axisymmetric load. The new material models are actually nonlinear stress-strain relations in an iteration procedure consistent with the new deformation theory of orthotropic plasticity described by Jones and Nelson.[2] The new models are qualified by comparison of predicted strains with strains measured by Jortner[1,6-9] for uniaxial off-axis loading and biaxial loading. He subjected graphite rods and bars to uniaxial loading in nonprincipal material directions and hollow tubular specimens to biaxial loading (combinations of axial tension or compression and internal pressure).

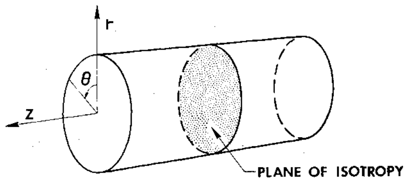

Fig. 1 Graphite billet coordinate system.

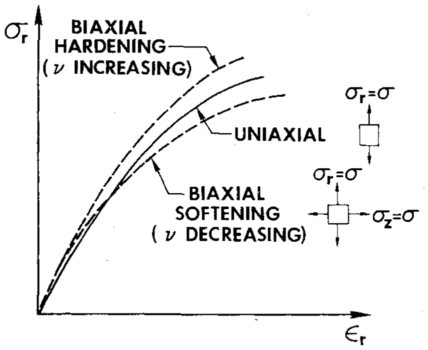

Fig. 2 Biaxial tension behavior of graphite (ν decreasing).

Received April 7, 1975; presented as part of Paper 75-769 at the AIAA/ASME/SAE 16th Structures, Structural Dynamics, and Materials Conference, Denver, Colo., May 27-29,1975; revision received November 14, 1975. This research was sponsored by Air Force Materials Laboratory (Contract F33615-73-C-5124) and Air Force Office of Scientific Research/NA (Grant 73-2532).

Index categories: Materials, Properties of; Structural Composite Materials (including Coatings); Structural Static Analysis.

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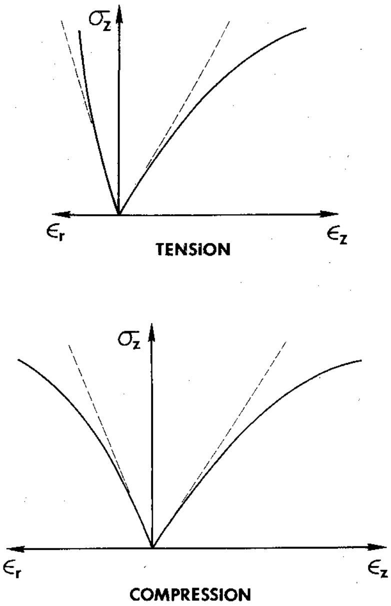

Fig. 3 Uniaxial stress-strain behavior of graphite under tension and compression.

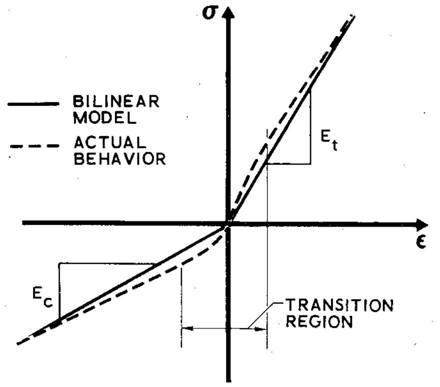

Fig. 4 Comparison of actual stress-strain behavior with the bilinear model

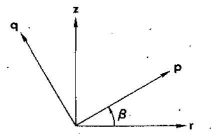

Fig. 5 Relation of principal stress (p-q-θ) coordinates to body and principal material (r-z-θ) coordinates.

The linear and nonlinear material models for different stress-strain behavior in tension and compression are described first. Then, in a companion paper,[10] the measured and predicted strains under uniaxial off-axis loading and under biaxial loading are correlated.

Elastic Models for Different Moduli in Tension and Compression

Introduction

Two material models for elastic behavior of materials with different moduli under tensile loading than under compressive loading are described in this section. In both models, the compliances S_{ij} in the strain-stress relations for axisymmetric bodies under axisymmetric loading

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} S_{11}^{rz} & S_{12}^{rz} & S_{13}^{rz} & S_{16}^{rz} \\ S_{12}^{rz} & S_{22}^{rz} & S_{23}^{rz} & S_{26}^{rz} \\ S_{13}^{rz} & S_{23}^{rz} & S_{33}^{rz} & S_{36}^{rz} \\ S_{16}^{rz} & S_{26}^{rz} & S_{36}^{rz} & S_{66}^{rz} \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} \tag{1}$$

are determined based on the principal stress state. In principal stress coordinates, the strain-stress relations are

$$\begin{Bmatrix} \epsilon_p \\ \epsilon_q \\ \epsilon_\theta \\ \gamma_{pq} \end{Bmatrix} = \begin{bmatrix} S_{11}^{pq} & S_{12}^{pq} & S_{13}^{pq} & S_{16}^{pq} \\ S_{12}^{pq} & S_{22}^{pq} & S_{23}^{pq} & S_{26}^{pq} \\ S_{13}^{pq} & S_{23}^{pq} & S_{33}^{pq} & S_{36}^{pq} \\ S_{16}^{pq} & S_{26}^{pq} & S_{36}^{pq} & S_{66}^{pq} \end{bmatrix} \begin{Bmatrix} \sigma_p \\ \sigma_q \\ \sigma_\theta \\ 0 \end{Bmatrix} \tag{2}$$

Note that 1) the θ direction is always a principal stress direction because of axisymmetric loading and 2) principal stress directions and principal strain coordinates do not coincide for orthotropic materials as they do for isotropic materials. Principal stress (p-q-θ) coordinates and body (r-z-θ) coordinates are shown in Fig. 5 (the θ direction is out of Fig. 5 toward the reader). Stresses, strains, and material properties will be transformed between these coordinate systems. Note that by definition body coordinates coincide with principal material coordinates for graphite.

The objective in both material models is to define a rational procedure for assigning the S_{ij}^{pq} on the basis of the principal stress state, the values of the S_{ij}^{pq} under tensile loading, and those values under compressive loading. The material properties depend on the stress state and vice versa; hence, the basic problem is statically indeterminate. However, the indeterminancy can be resolved by an apparently convergent iteration procedure consisting of four steps. First, displacements and stresses are calculated based on an initial assumption of stress signs with an implied initial choice of material properties. Second, the appropriate material properties are selected based on the principal stresses calculated in the previous step. Third, displacements and stresses including the new principal stresses are recalculated. Fourth, steps two and three are repeated until convergence to the desired accuracy is achieved.

The first model, called the weighted compliance matrix (WCM) model, is due to Jones.[4] This model consists basically of adding the tension and compression compliances in proportion to the presence of the respective tensile and compressive principal stresses; hence, the name weighted compliance matrix model is used. No theoretical basis exists for the weighted compliance matrix model; it is an engineering approach, rather than a scientific approach, to a very difficult problem.

The second model, called the restricted compliance matrix (RCM) model, is due mainly to Isabekian and Khachatryan[5] (modifications necessary to extend Isabekian and Khachatryan's model from a plane stress state to an axisymmetric stress state are described herein). This model has a more scientific basis than the weighted compliance matrix model. Specifically, the compliance matrix is made symmetric by prescribing certain relations between the tension and compression properties so that they satisfy the known transformations of anisotropic elasticity. However, by enforcing these relations between tension and compression properties, we limit our ability to treat real engineering materials.

Both models are attempts to treat materials with different moduli in tension and compression as special anisotropic elastic materials with the compliance matrix symmetry characteristics of ordinary elastic materials (for which a potential function exists). This compliance matrix symmetry is a correction to the basic Ambartsumyan model with a nonsymmetric compliance matrix.[11] This correction is necessary if anisotropic elasticity theory is to be used once the compliances are determined. Both material models are described and contrasted in the following sections.

Weighted Compliance Matrix Model

The following strain-stress relations in principal stress (p-q-θ) coordinates are extended for an axisymmetric body under axisymmetric load from plane stress relations proposed by Jones[4] for orthotropic materials that exhibit different moduli in tension and compression:

$$\begin{Bmatrix} \epsilon_p \\ \epsilon_q \\ \epsilon_\theta \\ \gamma_{pq} \end{Bmatrix} = \begin{bmatrix} S_{11}^{pq} & S_{12}^{pq} & S_{13}^{pq} & S_{16}^{pq} \\ S_{12}^{pq} & S_{22}^{pq} & S_{23}^{pq} & S_{26}^{pq} \\ S_{13}^{pq} & S_{23}^{pq} & S_{33}^{pq} & S_{36}^{pq} \\ S_{16}^{pq} & S_{26}^{pq} & S_{36}^{pq} & S_{66}^{pq} \end{bmatrix} \begin{Bmatrix} \sigma_p \\ \sigma_q \\ \sigma_\theta \\ 0 \end{Bmatrix} \quad (3)$$

Note again that the principal stress directions do not coincide with the principal strain directions. The compliances S_{ij}^{pq} are assigned according to the signs and magnitudes of the principal stresses:

if $\sigma_p > 0$; $S_{11}^{pq} = S_{11t}^{pq}$ (4a)

if $\sigma_p < 0$; $S_{11}^{pq} = S_{11c}^{pq}$ (4b)

if $\sigma_q > 0$; $S_{22}^{pq} = S_{22t}^{pq}$ (4c)

if $\sigma_q < 0$; $S_{22}^{pq} = S_{22c}^{pq}$ (4d)

if $\sigma_\theta > 0$; $S_{33}^{pq} = S_{33t}^{pq}$ (4e)

if $\sigma_\theta < 0$; $S_{33}^{pq} = S_{33c}^{pq}$ (4f)

if $\sigma_p > 0$ and $\sigma_q > 0$; $S_{12}^{pq} = S_{12t}^{pq}$ (4g)

if $\sigma_p < 0$ and $\sigma_q < 0$; $S_{12}^{pq} = S_{12c}^{pq}$ (4h)

if $\sigma_p > 0$ and $\sigma_q < 0$; $S_{12}^{pq} = k_{ppq} S_{12t}^{pq} + k_{qpq} S_{12c}^{pq}$ (4i)

if $\sigma_p < 0$ and $\sigma_q > 0$; $S_{12}^{pq} = k_{ppq} S_{12c}^{pq} + k_{qpq} S_{12t}^{pq}$ (4j)

if $\sigma_p > 0$ and $\sigma_\theta > 0$; $S_{13}^{pq} = S_{13t}^{pq}$ (4k)

if $\sigma_p < 0$ and $\sigma_\theta < 0$; $S_{13}^{pq} = S_{13c}^{pq}$ (4l)

if $\sigma_p > 0$ and $\sigma_\theta < 0$; $S_{13}^{pq} = k_{pp\theta} S_{13t}^{pq} + k_{\theta p\theta} S_{13c}^{pq}$ (4m)

if $\sigma_p < 0$ and $\sigma_\theta > 0$; $S_{13}^{pq} = k_{pp\theta} S_{13c}^{pq} + k_{\theta p\theta} S_{13t}^{pq}$ (4n)

if $\sigma_q > 0$ and $\sigma_\theta > 0$; $S_{23}^{pq} = S_{23t}^{pq}$ (4o)

if $\sigma_q < 0$ and $\sigma_\theta < 0$; $S_{23}^{pq} = S_{23c}^{pq}$ (4p)

if $\sigma_q > 0$ and $\sigma_\theta < 0$; $S_{23}^{pq} = k_{qq\theta} S_{23t}^{pq} + k_{\theta q\theta} S_{23c}^{pq}$ (4q)

if $\sigma_q < 0$ and $\sigma_\theta > 0$; $S_{23}^{pq} = k_{qq\theta} S_{23c}^{pq} + k_{\theta q\theta} S_{23t}^{pq}$ (4r)

if $\sigma_p > 0$; $S_{16}^{pq} = S_{16t}^{pq}$ (4s)

if $\sigma_p < 0$; $S_{16}^{pq} = S_{16c}^{pq}$ (4t)

if $\sigma_q > 0$; $S_{26}^{pq} = S_{26t}^{pq}$ (4u)

if $\sigma_q < 0$; $S_{26}^{pq} = S_{26c}^{pq}$ (4v)

if $\sigma_\theta > 0$; $S_{36}^{pq} = S_{36t}^{pq}$ (4w)

if $\sigma_\theta < 0$; $S_{36}^{pq} = S_{36c}^{pq}$ (4x)

where

$$k_{ppq} = |\sigma_p|/(|\sigma_p| + |\sigma_q|) \quad k_{qpq} = |\sigma_q|/(|\sigma_p| + |\sigma_q|) \quad (5a)$$

$$k_{pp\theta} = |\sigma_p|/(|\sigma_p| + |\sigma_\theta|) \quad k_{\theta p\theta} = |\sigma_\theta|/(|\sigma_p| + |\sigma_\theta|) \quad (5b)$$

$$k_{qq\theta} = |\sigma_q|/(|\sigma_q| + |\sigma_\theta|) \quad k_{\theta q\theta} = |\sigma_\theta|/(|\sigma_q| + |\sigma_\theta|) \quad (5c)$$

The weighting factors k_{ppq}, etc., are used to make the compliance matrix symmetric. Some other function of the principal stresses could be used to define the weighting factors. Full qualification of the form of the weighting factors awaits definitive experimental work. Note that only two of the three principal stresses are used to determine each of the cross-compliances S_{12}^{pq}, S_{13}^{pq}, and S_{23}^{pq}. Furthermore, a single principal stress is used to determine each of the cross-compliances S_{16}^{pq}, S_{26}^{pq}, and S_{36}^{pq}. The compliance S_{66}^{pq} cannot be rationally defined nor is S_{66}^{pq} necessary for transformation to any other coordinate system. The reason for the perhaps surprising lack of importance of S_{66}^{pq} is that it vanishes identically from all transformation relations *from* principal stress coordinates *to* any other coordinates.

The compliances S_{ijt}^{pq} and S_{ijc}^{pq} in the principal stress $(p-q)$ coordinates are related to the orthotropic compliances S_{ijt}^{rz} and S_{ijc}^{rz} in the principal material and body $(r-z)$ coordinates by the usual transformations of anisotropic elasticity:[12]

$$S_{11tc}^{pq} = S_{11tc}^{rz} \cos^4\beta + (2S_{12tc}^{rz} + S_{66tc}^{rz}) \sin^2\beta \cos^2\beta + S_{22tc}^{rz} \sin^4\beta \quad (6a)$$

$$S_{12tc}^{pq} = S_{12tc}^{rz} + (S_{11tc}^{rz} + S_{22tc}^{rz} - 2S_{12tc}^{rz} - S_{66tc}^{rz}) \sin^2\beta \cos^2\beta \quad (6b)$$

$$S_{13tc}^{pq} = S_{13tc}^{rz} \cos^2\beta + S_{23tc}^{rz} \sin^2\beta \quad (6c)$$

$$S_{16tc}^{pq} = (S_{66tc}^{rz} - 2S_{11tc}^{rz} + 2S_{12tc}^{rz}) \cos^3\beta \sin\beta - (S_{66tc}^{rz} - 2S_{22tc}^{rz} + 2S_{12tc}^{rz}) \sin^3\beta \cos\beta \quad (6d)$$

$$S_{22tc}^{pq} = S_{11tc}^{rz} \sin^4\beta + (2S_{12tc}^{rz} + S_{66tc}^{rz}) \sin^2\beta \cos^2\beta + S_{22tc}^{rz} \cos^4\beta \quad (6e)$$

$$S_{23tc}^{pq} = S_{13tc}^{rz} \sin^2\beta + S_{23tc}^{rz} \cos^2\beta \quad (6f)$$

$$S_{26tc}^{pq} = (S_{66tc}^{rz} - 2S_{11tc}^{rz} + 2S_{12tc}^{rz}) \sin^3\beta \cos\beta \quad (6g)$$

$$- (S_{66tc}^{rz} - 2S_{22tc}^{rz} + 2S_{12tc}^{rz}) \sin\beta \cos^3\beta \quad (6h)$$

$$S_{33tc}^{pq} = S_{33tc}^{rz} \quad (6i)$$

$$S_{36tc}^{pq} = 2(S_{23tc}^{rz} - S_{13tc}^{rz}) \sin\beta \cos\beta \quad (6j)$$

where the subscript t or c is taken as appropriate, and β is the angle between the body $(r\text{-}z\text{-}\theta)$ and principal material coordinates and the principal stress $(p\text{-}q\text{-}\theta)$ coordinates as defined in Fig. 5.

The compliances in principal material coordinates S_{ijtc}^{rz} are related to the engineering constants (direct moduli, Poisson's ratios, and shear moduli) by:

$$S_{11tc}^{rz} = 1/E_{rtc} \quad (7a)$$

$$S_{12tc}^{rz} = -\nu_{rztc}/E_{rtc} = -\nu_{zrtc}/E_{ztc} \quad (7b)$$

$$S_{13tc}^{rz} = -\nu_{r\theta tc}/E_{rtc} = -\nu_{\theta rtc}/E_{\theta tc} \quad (7c)$$

$$S_{22tc}^{rz} = 1/E_{ztc} \quad (7d)$$

$$S_{23tc}^{rz} = -\nu_{z\theta tc}/E_{ztc} = -\nu_{\theta ztc}/E_{\theta tc} \quad (7e)$$

$$S_{33tc}^{rz} = 1/E_{\theta tc} \quad (7f)$$

$$S_{66tc}^{rz} = 1/G_{rztc} \quad (7g)$$

where $\nu_{rzt} = -\epsilon_z/\epsilon_r$ for $\sigma_r = \sigma_t$ and all other stresses are zero. Apparently, seven independent material properties exist in tension in Eqs. (7) and the same number in compression. However, the compliances S_{66t}^{rz} and S_{66c}^{rz} ($1/G_{rzt}$ and $1/G_{rzc}$, respectively) cannot be measured in a shear test on an orthotropic material with different moduli in tension and compression since one principal stress is tension and the other is compression. Instead, in accordance with a suggestion by Tsai,[13] the tension modulus at 45° to the principal material axes E_t^{45} is measured and S^{rz}_{66t} is obtained from

$$S^{rz}_{66t} = 1/G_{rzt} = 4/E^{45}_t - (1/E_{rt} + 1/E_{zt} - 2\nu_{rzt}/E_{rt}) \quad (8)$$

A similar relation is used to define S^{rz}_{66c} in terms of E_c^{45}.

For axisymmetrically loaded axisymmetric transversely isotropic bodies, $E_{rtc} = E_{\theta tc}$ and $\nu_{rztc} = \nu_{\theta ztc}$. Thus, a possible set of independent material properties are

$$E_{rtc}, \nu_{rztc}, \nu_{r\theta tc}, E_{ztc}, E^{45}_{tc} \quad (9)$$

i.e., five properties in tension and five in compression for a total of ten independent properties.

Restricted Compliance Matrix Model

Isabekian and Khachatryan[5] define a symmetric compliance matrix for orthotropic materials with different moduli in tension and compression. They require the material properties in principal material directions to have interrelationships such that the compliances are symmetric in any coordinate system. These relationships are obtained in addition to the usual Ambartsumyan approach of superposing stress states of uniaxial tension in one direction and uniaxial compression in the other direction.

For example, if $\sigma_p > 0$ and $\sigma_q < 0$, then the strain-stress relations are

$$\epsilon_p = S^{pq}_{11t}\ \sigma_p + S^{pq}_{12c}\ \sigma_q \quad (10a)$$

$$\epsilon_q = S^{pq}_{12t}\ \sigma_p + S^{pq}_{22c}\ \sigma_q \quad (10b)$$

$$\gamma_{pq} = S^{pq}_{16t}\ \sigma_p + S^{pq}_{26c}\ \sigma_q \quad (10c)$$

For these relations to be symmetric,

$$S^{pq}_{12t} = S^{pq}_{12c} \quad (11)$$

If the compliance matrix is to be symmetric in principal material (r-z) directions, then

$$\nu_{rzt}/E_{rt} = \nu_{rzc}/E_{rc} = \nu_{zrt}/E_{zt} = \nu_{zrc}/E_{zc} \quad (12)$$

If, moreover, Eq. (11) is to be valid in all coordinate systems, then

$$1/E_{rt} - 1/E_{rc} = 1/E_{zt} - 1/E_{zc} = 1/E_t^{45} - 1/E_c^{45} \quad (13)$$

in which E_t^{45} and E_c^{45} are the tension and compression moduli, respectively, at 45° to the principal material directions. Note that E_t^{45} and E_c^{45} are related to the shear moduli in all tension or all compression stress states by Eq. (8) and its compression equivalent. Accordingly, the five independent material properties are

$$E_{rt}, E_{zt}, \nu_{rzt}, E_t^{45}, E_{rc} \quad (14)$$

or

$$E_{rc}, E_{zc}, \nu_{rzc}, E_c^{45}, E_{rt} \quad (15)$$

or other appropriate combinations of the foregoing relations.

For axisymmetric bodies under axisymmetric loads, the foregoing properties for the plane stress case must be supplemented by additional material properties. Because of the two kinds of axial symmetry, no additional shear modulus - related properties are necessary. However, the direct moduli $E_{\theta tc}$ must be defined in addition to the Poisson's ratios $\nu_{r\theta tc}$ and $\nu_{z\theta tc}$. The θ direction is automatically a principal stress direction. Thus, S_{33} in any coordinate system is affected only by $E_{\theta tc}$ and no other moduli because of the simplicity of the transformation relation. The cross-compliances S_{13} and S_{23} are symmetric under mixed tensile and compressive loading in principal material coordinates if

$$\nu_{r\theta t}/E_{rt} = \nu_{r\theta c}/E_{rc} \qquad \nu_{z\theta t}/E_{zt} = \nu_{z\theta c}/E_{zc} \quad (16)$$

However, E_{rt}, E_{rc}, E_{zt}, and E_{zc} are either independent properties or dependent and determined from Eq. (13). Thus, only two Poisson's ratios are independent, say $\nu_{r\theta t}$ and $\nu_{z\theta t}$. All transformations of stresses to obtain principal stresses are rotations *about* the θ direction. Hence, no conditions are imposed on $E_{\theta t}$ and $E_{\theta c}$. That is, no relation like Eq. (13) exists between $E_{\theta t}$ and $E_{\theta c}$. Accordingly, nine independent material properties are required for analyses of axisymmetrically loaded axisymmetric solids made of orthotropic multimodulus materials. Five of these properties come from the plane stress model and four are from the extension to axisymmetric solids. For transversely isotropic materials like ATJ-S graphite with the plane of isotropy in the r-θ plane, $E_\theta = E_r$, and $\nu_{rz} = \nu_{\theta z}$ etc. With these reductions, the six independent material properties are

$$E_{rt}, E_{zt}, \nu_{rzt}, \nu_{r\theta t}, E_t^{45}, E_{rc} \quad (17)$$

or appropriate equivalents through the foregoing relations.

The shear modulus for pure shear in coordinates 45° from the principal material coordinates can be shown to be independent of the sign of the shear stress for the restricted compliance matrix model. Consider, for example, the elements from a unidirectionally reinforced composite material shown in Fig. 6. There, the positive shear stress leads to tensile principal stress in the fiber direction and compressive principal stress in the direction transverse to the fibers. Similarly, the negative shear stress leads to compressive-stress in the fiber direction and tensile stress in the transverse direction. Because the tension and compression moduli are different in the fiber and transverse directions, the shear moduli at 45° would seem intuitively to be different. However, because of the relation between tension and compression moduli in Eq. (13), the expressions for the shear modulus at 45° under positive shear stress can be shown to be identical to the expression for the shear modulus at 45° under negative shear stress.

Summary

Irrespective of the material model used, the strain-stress relations in principal stress coordinates must be transformed to the body (r-z-θ) coordinates for solution of equilibrium

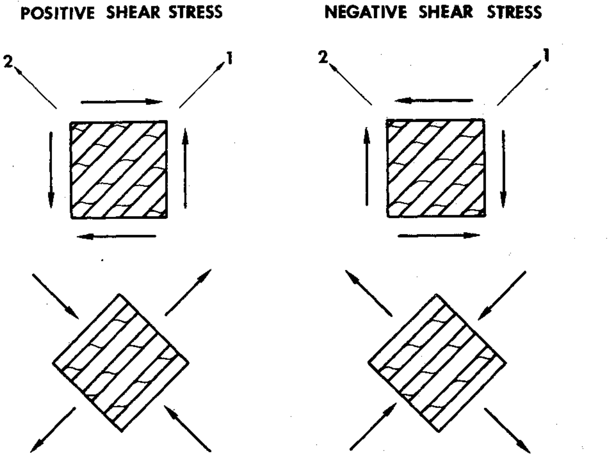

Fig. 6 Positive and negative shear stress at 45° to principal material directions.

problems. Those transformations take place according to the usual transformation relations of anisotropic elasticity (S_{66}^{pq} is omitted since it does not affect any strains):

$$S_{11}^{rz}=S_{11}^{pq}\cos^4\beta + 2S_{12}^{pq}\sin^2\beta\cos^2\beta+S_{22}^{pq}\sin^4\beta - (S_{16}^{pq}\cos^2\beta+S_{26}^{pq}\sin^2\beta)\sin 2\beta \tag{18a}$$

$$S_{12}^{rz}=S_{12}^{pq} + (S_{11}^{pq}+S_{22}^{pq} -2S_{12}^{pq})\sin^2\beta\cos^2\beta-\frac{1}{2}(S_{26}^{pq} - S_{16}^{pq})\sin 2\beta \cos 2\beta \tag{18b}$$

$$S_{16}^{rz}=-[S_{22}^{pq}\sin^2\beta-S_{11}^{pq}\cos^2\beta + S_{12}^{pq}\cos 2\beta]\sin 2\beta +S_{16}^{pq}\cos^2\beta(\cos^2\beta-3\sin^2\beta)+S_{26}^{pq}\sin^2\beta(3\cos^2\beta-\sin^2\beta) \tag{18c}$$

$$S_{22}^{rz}=S_{11}^{pq}\sin^4\beta + 2S_{12}^{pq}\sin^2\beta\cos^2\beta + S_{22}^{pq}\cos^4\beta + (S_{16}^{pq}\sin^2\beta + S_{26}^{pq}\cos^2\beta)\sin 2\beta \tag{18d}$$

$$S_{26}^{rz}=-[S_{22}^{pq}\cos^2\beta-S_{11}^{pq}\sin^2\beta-2S_{12}^{pq}\cos 2\beta]\sin 2\beta +S_{16}^{pq}\sin^2\beta(3\cos^2\beta-\sin^2\beta)+S_{26}^{pq}\cos^2\beta(\cos^2\beta-3\sin^2\beta) \tag{18e}$$

$$S_{66}^{rz}=4(S_{11}^{pq} + S_{22}^{pq}-2S_{12}^{pq})\sin^2\beta\cos^2\beta -2(S_{26}^{pq}-S_{16}^{pq})\sin 2\beta \cos 2\beta \tag{18f}$$

Moreover, stresses and displacements from both models for two distinct problems cannot be superimposed as can solutions for linear elastic problems. Superposition is invalid because the principal stress directions for two sets of stresses on the same body are generally different. However, if the principal stress directions are identical in both solutions, then the solutions can be superimposed. Ambartsumyan[11] calls this general invalidity of superposition a form of nonlinearity. In practical terms, each problem must be separately solved, and solutions cannot be built up by superposition of simple solutions.

The two material models described in this section have different numbers of independent material properties. The restricted compliance matrix model has six independent properties for an axisymmetrically loaded axisymmetric transversely isotropic body. For the same body, the weighted compliance matrix model has ten independent properties. Both models have advantages and disadvantages. The restricted compliance matrix model is derivable on a rational scientific basis; however, the compliance restrictions, as will be seen in the off-axis uniaxial loading correlation studies of the companion paper, are too stringent for many real materials. On the other hand, the weighted compliance matrix model is not derivable because the weighting factors are somewhat arbitrary; however, this model has the flexibility in "acceptable" compliances to be applicable to many real materials. Thus, the weighted compliance matrix model is an engineering approximation whereas the restricted compliance matrix model is a more scientific description of the basic phenomenon, but is restricted to a more limited class of materials.

Nonlinear Deformation Models for Different Moduli in Tension and Compression

Introduction

The objective of this section is to simultaneously model the nonlinear tension and compression stress-strain behavior of graphite shown in Fig. 3. The tension behavior is typical of a material that softens under increasing stress. That, is, among other characteristics, the Poisson's ratios decrease as the stress increases leading in part to the softening behavior illustrated in Fig. 2. On the other hand, the compression behavior is more nearly characteristic of a hardening

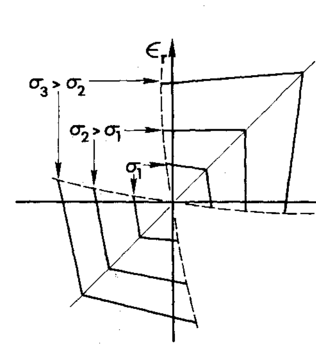

Fig. 7 Strain profiles under increasing stress levels for a material that softens under tension and hardens under compression.

material. That is, the Poisson's ratios are constant or increase as the stress increases. Thus, we might expect the hardening behavior in Fig. 2. Accordingly, strain profiles for various increasing stress levels might exist as schematically shown in Fig. 7 with biaxial tension in the upper right-hand quadrant and biaxial compression in the lower left-hand quadrant. However, no biaxial compression data exist to verify this speculation.

The contrasting tension and compression behavior illustrated in Fig. 7 may be perfectly realistic when the deformation mechanisms in tension are compared with the mechanisms in compression. Specifically, microcracking in tension may lead to an apparent increase in volume (decreasing Poisson's ratios). However, in compression, microcracking may not occur. The microcracking phenomenon may be related to the porous nature of graphite. The pores tend to open and perhaps tear under tensile stress. On the other hand, the pores tend to close and perhaps collapse under compression. Thus, graphite and other porous nonmetallic materials can have plastic volume changes when subjected to tensile or compressive stresses. Under compression, porous materials can compact (decrease in volume) whereas, under tension, they can dilate (increase in volume). Thus, classical plasticity theories with the usual zero plastic volume change hypothesis are not applicable to porous materials.

The nonlinear material model described by Jones and Nelson[2,3] is combined in this section with the linear multimodulus material models due to Jones[4] and to Isabekian and Khachatryan[5] to obtain nonlinear multimodulus material models. Thus, both the nonlinear and multimodulus deformation behavior of an orthotropic material can be modeled simultaneously, a necessity for describing ATJ-S graphite behavior.

The basic approach to analysis of nonlinear stress-strain behavior is a new deformation theory of orthotropic plasticity due to Jones and Nelson.[2] The various secant moduli and Poisson's ratios in the orthotropic stress-strain relations are approximated by

$$\text{Material Property}_i = A_i[1-B_i(U/U_{0_i})^{C_i}] \tag{19}$$

where the A_i are the elastic values of the material property, the B_i and C_i are related to the initial curvature and rate of change of curvature, respectively, of the stress-strain curve[3] (slightly different interpretations exist when the material property is a Poisson's ratio), and U is the strain energy density of an equivalent elastic system at each stage of nonlinear deformation:

$$U=(\sigma_r\epsilon_r+\sigma_z\epsilon_z+\sigma_\theta\epsilon_\theta+\tau_{rz}\gamma_{rz})/2 \tag{20}$$

The strain energy density U is normalized by U_{0_i} in Eq. (19) so that B_i and C_i are dimensionless.

The nonlinear stress-strain model is actually much more complicated than represented by Eq. (19). When mixed tensile and compressive stresses are considered, the strain energy used in Eq. (19) could be a weighted combination of the strain energy of compression and that of tension. Moreover, all coefficients have different values in tension than in compression. The choice of which properties, tension or compression, should be used is made in the Ambartsumyan superposition manner[11] after rotating the stress-strain relations to principal stress directions.

Two nonlinear multimodulus models with the same basic approach to nonlinear behavior are considered. Thus, the distinction between the models is based solely on the multimodulus formulations or, more specifically, on the determination of the compliance matrix. The linear multimodulus approach developed by Jones[4] is combined with the nonlinear approach due to Jones and Nelson[2] to obtain the "nonlinear weighted compliance matrix" (WCM) model. Isabekian and Khachatryan's extension[5] of Ambartsumyan's linear multimodulus approach[11] is combined with the nonlinear approach to obtain the "nonlinear restricted compliance matrix" (RCM) model. Nelson[15] discusses the nonlinear multimodulus models in more detail. The general characteristics of the iteration procedure common to both nonlinear multimodulus material models are discussed first. Then, the energy functions used to relate the multiaxial stress state to the material properties are described. Next, the distinctive characteristics of the nonlinear restricted compliance material model and the nonlinear weighted compliance model are discussed.

Iteration Procedure for Material Models

The overall iteration procedure common to both nonlinear multimodulus material models will be described. Basically, an indeterminate system of equations is solved with an iterative approach. The stresses and strains depend on the material properties which, in turn, depend on the stresses and strains. In the nonlinear approach due to Jones and Nelson,[2] the material properties are related to the strain energy density, which is the product of the stresses and the respective strains. In linear multimodulus models, the composition of the compliance matrix, hence the material properties, is determined from the signs and proportions of the principal stresses. In nonlinear multimodulus material models, the selection of material properties and, consequently, the stress-strain relationships is based on both the proportions of the principal stresses and the magnitude of an energy function.

The iteration procedure devised to simultaneously satisfy the constraints of both the nonlinear and multimodulus problems is illustrated schematically in Fig. 8. Each step in the procedure is described in the following paragraphs for a single element of a finite element approach to stress analysis.

The first step in the iteration procedure is to determine the material property versus strain energy relationships independently in tension and in compression from available uniaxial data. Thus, sets of constants A, B, C, and U_0 are determined for each independent material property. Separate sets of constants are determined for the tension and compression representation of a material property variation with strain energy if both sets are input to the procedure as independent material properties. The number of independent properties required for an analysis is dependent on the symmetry of the body and loading, the material model used, and the degree of anisotropy of the material.

The iteration procedures is started with the linear or "elastic" components of the tension materials properties. The desired stresses and strains (including the principal stresses and strains) are computed. Next, the principal stresses and their orientation are required for the multimodulus formulations. The strain energies are then calculated from the stresses and strains by use of one of three strain energy approaches discussed in the next section.

New independent material properties are determined after the strain energy is evaluated. This revised set of material properties is used to formulate new all-tension and all-compression compliance matrices. How these matrices are found is determined by which multimodulus approach is used.

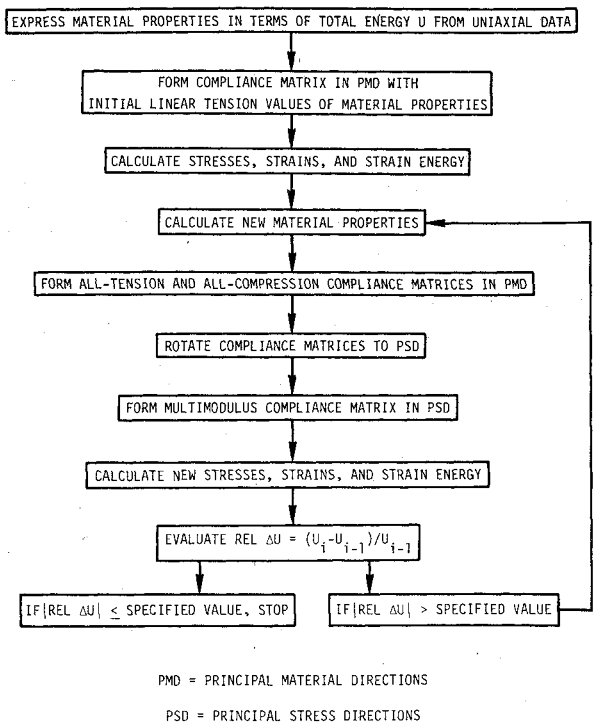

Fig. 8 Iteration procedure for nonlinear multimodulus materials.

Fewer material properties are required in the RCM model than in the WCM model for the same material because relationships exist in the RCM model between the all-tension and the all-compression compliances. The compliance matrices for the tensile and compressive stress states are then transformed to principal stress directions. Next, the composition of the multimodulus compliance matrix is determined by use of the signs of the principal stresses. The multimodulus compliance matrix is obtained from the transformed tension and compression compliance matrices by either the weighted compliance or restricted compliance matrix approach. After definition of a multimodulus compliance matrix, the stresses, strains, and strain energies are reevaluated. The loop from the calculation of new material properties to the computation of stresses, strains, and strain energies is repeated until the convergence criterion is satisfied.

A convergence criterion based on the relative change in total strain energy is compatible with both a nonlinear and a multimodulus material model and, therefore, is used in nonlinear multimodulus material models. Whenever the absolute value of the relative change in total strain energy, $|REL\ \Delta U|$, between two iterations $i-1$ and i

$$|REL\ \Delta U| = |(U_i - U_{i-1})/U_{i-1}| \tag{21}$$

becomes sufficiently small, then the iteration procedure is terminated and convergence is defined for a single element.

Convergence of the nonlinear multimodulus material models in finite element computer programs is defined to occur when the convergence criterion is simultaneously satisfied in each element. A finite element problem with a large number of elements can require a substantial amount of computer time for each iteration. Thus, the nonlinear multimodulus procedures are specifically devised to obtain rapid convergence by considering both the nonlinear and multimodulus characters of the problem on the same iteration.

Strain Energy Functions

Three different energy functions – total strain energy, divided strain energy, and weighted strain energy are investigated for the nonlinear multimodulus material models. Each of these functions will be discussed briefly. The strain energies used in the material models are actually strain energy densities. Although the word density is usually omitted, the strain energies are always understood to be on a unit volume basis.

The total strain energy defined in Eq. (20) can be used to determine all material properties in each iteration as is done in the nonlinear model of Jones and Nelson.[2] Thus, application of the total energy function is the same for both the nonlinear material model and the nonlinear multimodulus material models.

In the divided energy approach, the total strain energy is separated into two components: 1) the contribution from the tensile principal stresses and 2) the contribution from the compressive principal stresses. These two components are not invariant under rotation of coordinates and are defined only in principal stress coordinates. However, their sum, the total strain energy, is invariant with respect to coordinate transformations. In the divided energy approach, the tension component of strain energy is used to determine the tension material properties, and the compression component of strain energy is used to determine the compression material properties. This division of energy is a mechanism used to investigate dependence of the relationship between strain energy and material properties on the signs of the multiaxial principal stresses.

In the weighted energy approach, the effective energy level in terms of the tension and compression components of the total strain energy density is

$$U_w = \left(\frac{U_c}{U}\right)^n U_c + \left[1 - \left(\frac{U_c}{U}\right)^n\right] U_t \qquad (22)$$

where n = a positive integer constant, U = total strain energy, U_c = strain energy of compressive stresses, U_t = strain energy of tensile stresses, and U_w = weighted energy. This energy level U_w is used to find both the tension and the compression material properties. Thus, this approach is more desirable than the divided energy method from the standpoint that a consistent energy level is used to determine the current values of all material properties. When $n=1$ in Eq. (22), the weighted energy density varies between the average and the sum of U_t and U_c as a function of the ratio U_c/U. This variation is shown in Fig. 9 along with the variation for $n=2$. When $n=1$, the variation is symmetrical about $U_c/U=.5$. On the other hand, for $n=2$, the weighted energy density is larger when $(U_c:U_t)=(1:9)$ than when $(U_c:U_t)=(9:1)$. When the weighted energy is used in comparisons between predicted and measured strains,[10] n is always unity. Other values of n have not be investigated.

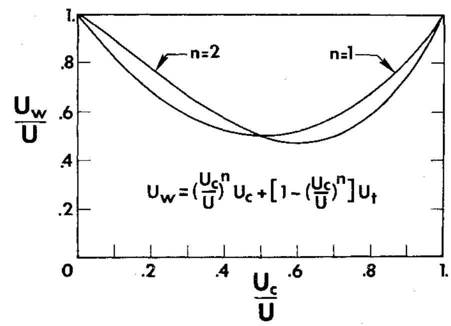

Fig. 9 Variation of weighted strain energy.

Nonlinear Weighted Compliance Matrix Model

Three aspects of the nonlinear weighted compliance material model (WCM) are discussed: 1) the distinctive features of the iteration approach, 2) the restrictions imposed on the stress-strain behavior that can be treated with the model, and 3) the computerized application of the material model.

Two steps in the general nonlinear multimodulus procedure outlined in Fig. 8 depend on which multimodulus procedure is used. The first step is to evaluate the all-tension and all-compression compliance matrices. The elements of these matrices are determined from the material properties by the same expressions in both the linear and nonlinear multimodulus approaches. The material properties are constant in the linear approach, but are functions of the strain energy in the nonlinear approach. The second step which depends on the multimodulus procedure is the computation of the multimodulus compliance matrix. This step is identical in both the linear and the nonlinear weighted compliance material models.

When the linear multimodulus approach is combined with the nonlinear approach, restrictions are placed on the deformation behavior of the material that can be treated. As with the nonlinear material model,[2] the reciprocal relations between material properties are assumed to be valid when all material properties are evaluated at the same energy level. Reciprocal relations exist for both tension and compression properties in the multimodulus problem. Symmetry of the multimodulus compliance matrix in each iteration is ensured in the nonlinear WCM model by weighting factors used in the development of the multimodulus compliance matrix, not by relationships between tension and compression material properties as will be done in the RCM model. Thus, with the nonlinear weighted compliance material model, fewer restrictions are placed on the stress-strain behavior that can be treated. However, in the application of the nonlinear WCM model, more independent tension and compression material properties are required than for the nonlinear RCM model. Also, restrictions on the material behavior for the reciprocal relations are the same for all three energy functions due to the independence of tension and compression compliances.

A short computer program, MULTIW, is used to investigate the weighted compliance model for all three energy functions. MULTIW is limited in capability to the prediction of strains from an input uniform stress state. However, MULTIW has a degree of generality compatible with the material models in a modified version of the SAAS III finite element program.[14] The stress-strain relations in MULTIW can be applied to an axisymmetric body under axisymmetric load as well as to more general stress states for isotropic, transversely isotropic, and orthotropic materials. For the latter two levels of anisotropy, the principal material (x, y, z) directions can be at arbitrary orientations in the y-z plane to the input stress (x', y', z') directions. The input stresses are constant so the principal stresses and their orientations do not vary from one iteration to the next. This program will be used in the off-axis strain response studies and the biaxial strain response studies.[10]

Nonlinear Restricted Compliance Matrix Material Model

Three aspects of the nonlinear restricted compliance matrix (RCM) material model are discussed: 1) the distinctive features of the iteration approach, 2) the restrictions imposed on the stress-strain behavior that can be treated with the model, and 3) the computerized application of the material model.

Only two steps in the general iteration procedure are affected by which multimodulus procedure is used. These steps involve the evaluation of the all-tension and all-compression compliance matrices in principal material coordinates and the evaluation of the multimodulus compliance matrix in principal stress directions. The linear multimodulus approach is used to develop these compliance matrices for the nonlinear restricted compliance material model. In this approach, the symmetry of the compliance matrices about the main diagonal in all coordinate systems is ensured by the relation between the material properties determined in uniaxial tension and uniaxial compression. In each iteration of the nonlinear procedure, a linear (elastic) system is considered which for the "last" iteration is equivalent to the nonlinear system. The requirement of symmetry of the compliance matrix is then related to the existence of a potential function for such a system.

The main difference between the linear and the nonlinear RCM model is the variation of material properties with strain energy. For the linear material model, all the input tension and compression material properties are constant. Therefore, the multimodulus compliance matrix is affected only by the signs of the principal stresses, not their magnitudes. In the nonlinear multimodulus material model, the material properties used to define the all-tension and all-compression compliance matrices are functions of the strain energy. In the first iteration, the linear tension material properties are used to predict the stresses and strains. However, in subsequent iterations, the current value of the strain energy (or energies in the case of the divided energy) is used to determine the material properties in the stress-strain relationship.

For the total or weighted energy approaches, the relationships between material properties are implied to be valid when all the material properties are evaluated at the same strain energy. Thus, restrictions are placed on the uniaxial stress-strain behavior that can be treated with the models. That is, not all materials have properties for which the material model restrictions are satisfied.

Even more restrictions are imposed on the deformation behavior of the material being investigated if the divided energy approach is used. In the nonlinear RCM model, the off-diagonal compliances in corresponding positions of the all-tension and all-compression compliance matrices are required to be identical at the same energy level. (This requirement may lead to limited applicability of the model.) If the divided energy approach is used, these off-diagonal compliances must be constant for all energy levels. A cube of an isotropic multimodulus material loaded only in two orthogonal directions by equal tensile forces can be used to illustrate the constancy requirement of the off-diagonal compliances. Suppose two of the three independent material properties for a nonlinear multimodulus isotropic material were determined in compression (despite the fact that we deal with an all-tension stress state). Then, the two compression properties would be used to evaluate all the nonzero off-diagonal compliances. The compression component of the total strain energy would be used to find the current value of the two compression properties. The compression component of total strain energy would always be zero so these compliances are constant. Thus, if the material is to satisfy the reciprocal relation

$$\nu_t/E_t = \nu_c/E_c \tag{23}$$

at all energy levels, the material property expressions for the Young's modulus and Poisson's ratio in tension could differ at most in the constant A. Therefore, the applicability of the nonlinear RCM model to actual deformation behavior must be assessed by examining whether (or how well) the relations between tension and compression compliances are satisfied.

A short computer program, MULTIR, is used to investigate the restricted compliance material model for all three energy functions. This program is identical in form and character to the MULTIW computer program described for the weighted compliance material model. Both programs will be used in the off-axis strain response studies and the biaxial strain response studies.[10]

Summary

The nonlinear material model is combined with each of the linear multimodulus material models to obtain two nonlinear multimodulus material models. The use of three different energy functions is investigated for both of the two models. Restrictions on the stress-strain behavior that can be treated are discussed.

The actual stress-strain behavior of a nonlinear multimodulus material would probably have the general form of the dashed curve in Fig. 4. In the linear multimodulus models, a bilinear approximation is represented by the solid lines in Fig. 4. The discontinuity in slope at the origin for the bilinear approximation would probably not occur for an actual material. Instead, a nonlinear transition region would be expected. This transition region can be represented with the nonlinear multimodulus material models by use of the same value of the material property constant A, but different values of B and C, in the tension and compression equations for a material property. For problems in which the stress-strain behavior in the transition region is not important, a better representation of the actual material property-strain energy variations might be achieved with different values of A in the tension and compression equations for a material property.

Concluding Remarks

New material models are described for nonlinear deformation behavior of artificial graphite under initial loading. The models are based on a new deformation theory of orthotropic plasticity[2] and include both nonlinear behavior and different behavior under tensile than compressive loads. Every independent material property can vary in an arbitrary manner and is related to the multiaxial state of stress and strain by a strain energy function. The signs of the principal stresses are used along with strain energy functions to determine the material properties. The properties are therefore a function of the energy which depends on stresses and strains which in turn depend on the material properties. Accordingly, an iteration procedure is used to simultaneously satisfy the nonlinear stress-strain relations and the material property versus strain energy equations. The iteration procedure is incorporated in a new version of a finite element stress analysis computer program.[14]

Two alternative approaches are explored for selection of properties in the strain-stress relations: 1) a weighted compliance matrix (WCM) model due to Jones[4] and 2) a restricted compliance matrix (RCM) model extended from the work of Isabekian and Khachatryan.[5] In the WCM model, the tension compliances are weighted with the compression compliances according to the proportion of tensile and compressive principal stresses to obtain the compliances in the strain-stress relations. On the other hand, in the RCM model, the tension compliances must be related to the compression compliances according to the compliance restrictions of Isabekian and Khachatryan (hence the name restricted compliance matrix model). Both of these models are extended to nonlinear deformation in the present paper.

The applicability and accuracy of the nonlinear multimodulus material models, when incorporated in a modified version of the SAAS III finite element computer program,[14] are investigated in the uniaxial off-axis strain response and biaxial strain response studies of the companion paper.[10]

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